SS 2016

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Differential Geometry

Homework 6

Mandatory Exercise 1. (10 points)
Let Z^k be the set of all closed k-forms on a smooth manifold M and define a relation $\sim$ between forms in Z^k as follows: $\alpha \sim \beta$ if and only if they differ by an exact form, that is, if $\beta - \alpha = d\theta$ for some $(k-1)$-form θ.

(a) This relation is an equivalence relation.

(b) Let $H^k(M)$ be the corresponding set of equivalence classes (called the k-dimensional **de Rham cohomology** vector space of M). Show that addition and scalar multiplication of forms define indeed a vector space structure on $H^k(M)$.

(c) Let $f\colon M \to N$ be a smooth map. Show that:

- The pull-back f^* carries closed forms to closed forms and exact forms to exact forms.
- If $\alpha \sim \beta$ on N then $f^*\alpha \sim f^*\beta$ on M.
- f^* induces a linear map on cohomology also called $f^*\colon H^k(N) \to H^k(M)$ naturally defined by $f^*[\omega] := [f^*\omega]$.
- If $g\colon L \to M$ is another smooth map, then $(f \circ g)^* = g^* \circ f^*$.

(d) Show that the dimension of $H^0(M)$ is equal to the number of connected components of M.

(d) Show that $H^k(M) = \{0\}$ for every $k > \dim(M)$.

Mandatory Exercise 2. (10 points)
Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. Show that for every regular value $a \in \mathbb{R}$ of f the gradient $\text{grad}(f)$ is orthogonal to the submanifold $f^{-1}(a)$.

Suggested Exercise 1. (0 points)
We say that a differentiable curve $\gamma\colon [a, b] \to M$ is obtained from the curve $c\colon [c, d] \to M$ by **reparametrization** if there exists a smooth bijection $f\colon [a, b] \to [c, d]$ (the reparametrization) such that $\gamma = c \circ f$. Show that the length of a curve does not change under reparametrization.

Suggested Exercise 2. (0 points)
Denote by $SO(n)$ the special orthogonal group consisting of orthogonal $n \times n$ matrices with determinant 1.

(a) Show that the two Lie groups $S^1 \subset \mathbb{C}$ and $SO(2)$ are isomorphic. (Here isomorphic means isomorpic as Lie groups, i.e. there exists a group isomorphism which is also a diffeomorphism.)

(b) Consider the two Lie groups $O(n)$ and $SO(n) \times \mathbb{Z}_2$. Are they homeomorphic? Are they isomorphic as Lie groups?

Suggested Exercise 3. (0 points)
Let S be a tensor of type $(0, 1)$, i.e. a section of T^*M, and X, Y be vector fields on M. Show that

$$(\mathcal{L}_X S)(Y) = X(S(Y)) - S([X, Y]).$$

Hint: Start from $X(S(Y))$ and view it as a Lie derivative of a function $S(Y)$ in direction of X.

Suggested Exercise 4. (0 points)
Let $g = \sum_{i,j=1}^n g_{ij}\, dx^i \otimes dx^j \in T_p^{(0,2)} M$ be a $(0, 2)$-tensor. Show that:

(a) g is symmetric if and only if $g_{ij} = g_{ji}$.

(b) g is non-degenerate if and only if $\det(g_{ij}) \neq 0$.

(c) g is positive definite if and only if (g_{ij}) is positive definite.

(d) If g is non-degenerate, the map $\Phi_g \colon T_pM \to T_p^*M$ given by $\Phi_g(v)(w) = g(v, w)$ is a linear isomorphism.

(e) If g is positive definite then g is non-degenerate.

Hand in: Monday 30th May
in the exercise session
in Seminar room 2, MI